MIDTERM: The midterm will be in class on Thursday February 12. The material will be everything up to that covered in the homework due on Feb. 10.

1. A cubic box of side $a$ has five faces at zero potential, and the sixth (the top face) is at a constant potential $V_0$, see Griffiths Fig. 3.23. Find the potential inside the box. What is the potential in the center? Hint: Think superposition.

2. Consider a conducting sphere of radius $a$ at potential $V_1$ at the center of a conducting shell of radius $b$ ($> a$) at potential $V_2$. Find the potential in between the conductors ($a < r < b$).

3. Consider the same problem as the previous one except that the charge is specified to be $Q$ on the inner sphere (rather than the potential being specified). Note: Because of symmetry you can deduce the surface charge density, $\sigma$, from the total charge. From $\sigma$ you can get $E_r$ at the surface, and this gives $\partial V / \partial r$. (Here the normal to the surface is in the $r$ direction.) Hence the boundary conditions are that $V$ is specified on the outer surface and $\partial V / \partial r$ is specified on the inner surface.

4. The potential at the surface of a sphere of radius $R$ is $V_0(\theta) = k \cos 3\theta$, where $k$ is a constant. Find the potential inside and outside the sphere as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume that there are no charges inside or outside the sphere; only on the surface.)

5. Solve Laplace’s equation by separation in cylindrical coordinates, assuming that there is no dependence on $z$ (cylindrical symmetry). Note: Make sure that you find all solutions to the radial equation; in particular your result must include the case of an infinitely long line charge, for which we have already discussed the result.
6. Show that the quadrupolar term in the multipole expansion can be written

\[ V_{\text{quad}}(r) = \frac{1}{4\pi \epsilon_0} \frac{1}{2r^3} \sum_{i,j=1}^{3} \hat{r}_i \hat{r}_j Q_{ij}, \]

where

\[ Q_{ij} \equiv \int [3r'_i r'_j - (r')^2 \delta_{ij}] \rho(r') \, d\tau', \]

and \( \delta_{ij} \) is the Kronecker delta function. \( Q_{ij} \) is called the \textbf{quadrupole moment} of the charge distribution. \( \]

\textit{Note:} The strength of successive terms in the multipole expansion is determined by the total charge \( Q \), which is a scalar, the dipole moment \( \mathbf{p} \), which is a vector, and the \( Q_{ij} \) which form a \textit{second rank tensor}, and so on.

7. Show that the total electric field \( \mathbf{E} \) integrated over the inside of a sphere of radius \( R \) due to all the charge within the sphere is given by

\[ \int_{\text{sphere}} \mathbf{E}(\mathbf{r}') \, d\tau' = -\frac{1}{3\epsilon_0} \mathbf{p}_{\text{tot}}, \]

where \( \mathbf{p}_{\text{tot}} \) is the total dipole moment \( \mathbf{p}_{\text{tot}} = \sum_i q_i \mathbf{r}_i \).

One way to derive this simple result is:

(a) Show that the average field due to a single charge \( q \) at a point \( \mathbf{r} \) inside a sphere is the same as the field at \( \mathbf{r} \) due to a uniformly charged sphere with \( \rho = -q/(4\pi R^3) \), namely

\[ \frac{1}{4\pi \epsilon_0} \frac{q}{4\pi R^3} \int \frac{1}{r^2} \hat{\mathbf{e}} \, d\tau', \]

where \( \hat{\mathbf{e}} = \mathbf{r}' - \mathbf{r} \) is the vector from \( \mathbf{r} \) to \( \mathbf{r}' \), and \( \mathbf{r}' \) is integrated over the sphere.

(b) The latter can be found using Gauss’s law. Express the answer in terms of the dipole moment of \( q \).

(c) Use the principle of superposition to generalize to an arbitrary charge distribution.

8. Do Griffiths 3.42.

In this question you show that the expression derived in class and in the book, Eq. (3.104), for a “pure” dipole,

\[ \mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi \epsilon_0} \frac{1}{r^3} \left[ 3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p} \right], \quad (1) \]

is incomplete because the derivation does not take proper account of divergence of \( \mathbf{E} \) at \( \mathbf{r} = 0 \). The question shows that, in fact, one needs to add to Eq. (1) the following term:

\[ -\frac{1}{3\epsilon_0} \mathbf{p} \delta^{(3)}(\mathbf{r}). \]

\textit{Note:} This is analogous to the case of the electric field of a point charge which naively has zero divergence but we showed by a more careful argument that there is a delta function contribution at the origin, i.e.

\[ \nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) = -4\pi \delta^{(3)}(\mathbf{r}) . \]